HOW DO CALCULATORS CALCULATE TRIGONOMETRIC FUNCTIONS?

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ABSTRACT

How does your calculator quickly produce the values of trigonometric functions? You might be surprised to learn that it does not use series or polynomial approximation, but rather the so-called CORDIC method. This paper will focus on the geometry of the CORDIC method, as originally developed by Volder in 1959. This algorithm is a wonderful application of sequences and will be demonstrated on the TI-86. We will also provide a rigorous convergence proof for the CORDIC method.

1. Introduction.

We presented a general introduction to the CORDIC method last year at the Ninth Annual ICTCM meeting in Reno [1]. In this paper we will focus on how the CORDIC algorithm is used on calculators for finding the values of the trigonometric functions.

Given an angle $\theta, -\pi/2 \leq \theta \leq \pi/2$, the CORDIC equations for approximating the sine and cosine of θ are

$$\begin{aligned} x_{k+1} &= x_k - \delta_k y_k 2^{-k} \\ y_{k+1} &= y_k + \delta_k x_k 2^{-k} \\ z_{k+1} &= z_k - \delta_k \tan^{-1} 2^{-k}. \end{aligned}$$

The constants $\sigma_k = \tan^{-1} 2^{-k}$ are permanently stored in the calculator's memory. The constants δ_k are nothing more than comparisons, given by the signum function

$$\delta_k = \operatorname{sgn}(z_k) = \begin{cases} 1, & z_k \geq 0 \\ -1, & z_k < 0, \end{cases}$$

and have the effect of driving the z values toward zero.

The starting values are $x_0 = K = \prod_{j=0}^{n} \cos(\sigma_j)$, a stored constant, $y_0 = 0$, and $z_0 = \theta$, the given angle. We will show that for large n, $z_{n+1} \approx 0$, $x_{n+1} \approx \cos\theta$, and $y_{n+1} \approx \sin\theta$. Notice that these equations involve only additions, subtractions, digit shifts (multiplication by 2^{-k}), comparisons, and the recall of a small number of stored constants $\sigma_k = \tan^{-1} 2^{-k}$. (Although the CORDIC algorithm is actually programmed in base 10, we will present the main ideas in binary arithmetic.)

The CORDIC algorithm was originally developed for calculating trigonometric functions [3]. The above equations were given the name *rotation* mode because the z values are "rotated" to zero. We will look closely at this geometric interpretation in Section 2.

2. Geometrical Interpretation of CORDIC.

We analyze the geometry of the CORDIC trigonometric algorithm by rewriting the equations for x and y in the following matrix form.

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & -\delta_k 2^{-k} \\ \delta_k 2^{-k} & 1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

Since $\sigma_k = \tan^{-1} 2^{-k}, \tan\sigma_k = 2^{-k}$ and $\sin\sigma_k = (\cos\sigma_k)2^{-k}$. Hence, we can rewrite the 2×2 matrix above as

$$\begin{aligned} \frac{1}{\cos\sigma_k} \begin{bmatrix} \cos\sigma_k & -\delta_k(\cos\sigma_k)2^{-k} \\ \delta_k(\cos\sigma_k)2^{-k} & \cos\sigma_k \end{bmatrix} &= \frac{1}{\cos\sigma_k} \begin{bmatrix} \cos\sigma_k & -\delta_k \sin\sigma_k \\ \delta_k \sin\sigma_k & \cos\sigma_k \end{bmatrix} \\ &= \frac{1}{\cos\sigma_k} \begin{bmatrix} \cos(\delta_k\sigma_k) & -\sin(\delta_k\sigma_k) \\ \sin(\delta_k\sigma_k) & \cos(\delta_k\sigma_k) \end{bmatrix}. \end{aligned}$$

This matrix can be interpreted as the matrix of a "scaled rotation." The matrix

$$\begin{bmatrix} \cos(\delta_k\sigma_k) & -\sin(\delta_k\sigma_k) \\ \sin(\delta_k\sigma_k) & \cos(\delta_k\sigma_k) \end{bmatrix}$$

produces a counterclockwise rotation through an angle $\delta_k\sigma_k = \pm\sigma_k = \pm\tan^{-1} 2^{-k}$ while the factor $1/(\cos\sigma_k)$ scales the result. Geometrically, the point (x_k, y_k) is rotated by an angle $\pm\sigma_k = \pm\tan^{-1} 2^{-k}$ and then stretched by the factor $1/(\cos\sigma_k)$, as indicated in Figure 1.

It is now easy to see why this algorithm works. From the equation $z_{k+1} = z_k - \delta_k \sigma_k = z_k - \delta_k \tan^{-1} 2^{-k}$ and $z_0 = \theta$, it follows that $z_{n+1} = \theta - \sum_{k=0}^{n} \delta_k \sigma_k$. For large n, we will show that $z_{n+1} \approx 0$ and $\theta \approx \sum_{k=0}^{n} \delta_k \sigma_k$. By composing the $(n+1)$ rotations, the following analysis shows that the x values tend to $\cos\theta$ and the y values tend to $\sin\theta$:

$$\begin{aligned}
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} &= \frac{1}{\cos\sigma_0} \begin{bmatrix} \cos(\delta_0\sigma_0) & -\sin(\delta_0\sigma_0) \\ \sin(\delta_0\sigma_0) & \cos(\delta_0\sigma_0) \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix} \\
\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} &= \frac{1}{\cos\sigma_0 \cos\sigma_1} \begin{bmatrix} \cos(\delta_0\sigma_0 + \delta_1\sigma_1) & -\sin(\delta_0\sigma_0 + \delta_1\sigma_1) \\ \sin(\delta_0\sigma_0 + \delta_1\sigma_1) & \cos(\delta_0\sigma_0 + \delta_1\sigma_1) \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix} \\
&\vdots \\
\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} &= \frac{1}{\cos\sigma_0 \cos\sigma_1 \cdots \cos\sigma_n} \begin{bmatrix} \cos(\sum_{k=0}^{n} \delta_k\sigma_k) & -\sin(\sum_{k=0}^{n} \delta_k\sigma_k) \\ \sin(\sum_{k=0}^{n} \delta_k\sigma_k) & \cos(\sum_{k=0}^{n} \delta_k\sigma_k) \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix} \\
&\approx \frac{1}{K} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} K \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}
\end{aligned}$$

3. Convergence.

The convergence properties of the sequences arising from the CORDIC algorithm depend on the following key theorem ([2], p. 320).

Convergence Theorem. *Let $\sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_n > 0$ be a decreasing sequence of positive numbers satisfying*

$$\sigma_k \leq \sigma_n + \sum_{j=k+1}^{n} \sigma_j, \quad \text{for } 0 \leq k < n.$$

Let r be a number satisfying

$$|r| \leq \sigma_n + \sum_{j=0}^{n} \sigma_j.$$

Define the sequence $s_0 = 0$ and $s_{k+1} = s_k + \rho_k \sigma_k$, $k = 0, 1, \ldots, n$, where

$$\rho_k = sgn(r - s_k) = \begin{cases} 1, & r \geq s_k \\ -1, & r < s_k. \end{cases}$$

Then

$$|r - s_k| \leq \sigma_n + \sum_{j=k}^{n} \sigma_j, \quad for \ \ 0 \leq k \leq n.$$

In particular, $|r - s_{n+1}| \leq \sigma_n$.

Proof. The proof is by induction on k. For $k = 0$, we immediately have

$$|r - s_0| = |r| \leq \sigma_n + \sum_{j=0}^{n} \sigma_j.$$

Now assume that the theorem is true for k and consider $|r - s_{k+1}| = |r - s_k - \rho_k \sigma_k|$. If $r - s_k \geq 0$, then $\rho_k = 1$ and we have $|r - s_k - \rho_k \sigma_k| = \Big|\, |r - s_k| - \sigma_k \,\Big|$. On the other hand, if $r - s_k < 0$, then $\rho_k = -1$ and $|r - s_k - \rho_k \sigma_k| = |r - s_k + \sigma_k| = |s_k - r - \sigma_k| = \Big|\, |r - s_k| - \sigma_k \,\Big|$. Therefore, in either case we have $|r - s_{k+1}| = |r - s_k - \rho_k \sigma_k| = \Big|\, |r - s_k| - \sigma_k \,\Big|$.

From the first inequality hypothesis we have

$$-\left(\sigma_n + \sum_{j=k+1}^{n} \sigma_j\right) \leq -\sigma_k \leq |r - s_k| - \sigma_k.$$

By the induction hypothesis,

$$|r - s_k| - \sigma_k \leq \left(\sigma_n + \sum_{j=k}^{n} \sigma_j\right) - \sigma_k = \left(\sigma_n + \sum_{j=k+1}^{n} \sigma_j\right).$$

Combining these two inequalities,

$$|r - s_{k+1}| = \Big|\, |r - s_k| - \sigma_k \,\Big| \leq \sigma_n + \sum_{j=k+1}^{n} \sigma_j,$$

which shows that the theorem is true for $k + 1$. Finally, $-\sigma_n \leq |r - s_n| - \sigma_n \leq 2\sigma_n - \sigma_n = \sigma_n$ and thus $|r - s_{n+1}| = \Big|\, |r - s_n| - \sigma_n \,\Big| \leq \sigma_n$, which completes the proof. ■

The following theorem states that the sequence $\sigma_k = \tan^{-1} 2^{-k}, k = 0, 1, \dots n$, used in the trigonometric algorithm, satisfies the Convergence Theorem.

Theorem. *For $n > 3$, the sequence $\sigma_k = \tan^{-1} 2^{-k}, k = 0, 1, \dots, n$ satisfies the hypothesis of the convergence theorem for all $|r| \le \pi/2$.*

Proof. The given sequence $\tan^{-1} 1 = \pi/4, \tan^{-1} \frac{1}{2}, \tan^{-1} \frac{1}{4}, \dots, \tan^{-1} \frac{1}{2^n}$ is clearly a decreasing sequence of positive numbers.

The Mean Value Theorem says that there exists a number c between a and b such that

$$\frac{\tan^{-1} b - \tan^{-1} a}{b - a} = \frac{1}{1 + c^2}, \qquad a < c < b.$$

Let $a = 2^{-(j+1)}$ and $b = 2^{-j}$ in this equation. Then $b - a = 2^{-(j+1)}$ and

$$\frac{1}{1+c^2} < \frac{1}{1+a^2} = \frac{1}{1+2^{-2j-2}} = \frac{2^{2j+2}}{1+2^{2j+2}}.$$

Hence,

$$\sigma_j - \sigma_{j+1} = (b-a)\frac{1}{1+c^2} \le \frac{1}{2^{j+1}} \frac{2^{2j+2}}{1+2^{2j+2}} = \frac{2^{j+1}}{1+2^{2j+2}}.$$

Now let $a = 0$ and $b = 2^{-j}$ in the Mean Value Theorem equation. Then we have

$$\frac{1}{1+c^2} > \frac{1}{1+b^2} = \frac{1}{1+2^{-2j}} = \frac{2^{2j}}{1+2^{2j}},$$

and

$$\sigma_j = b\,\frac{1}{1+c^2} \ge \frac{1}{2^j}\,\frac{2^{2j}}{1+2^{2j}} = \frac{2^j}{1+2^{2j}}.$$

We combine these inequalities involving the σ_j using a telescoping series.

$$\begin{aligned}\sigma_k - \sigma_n &= (\sigma_k - \sigma_{k+1}) + (\sigma_{k+1} - \sigma_{k+2}) + \cdots + (\sigma_{n-1} - \sigma_n) \\ &= \sum_{j=k}^{n-1} (\sigma_j - \sigma_{j+1}) \\ &\le \sum_{j=k}^{n-1} \frac{2^{j+1}}{1 + 2^{2j+2}} \\ &= \sum_{j=k+1}^{n} \frac{2^j}{1 + 2^{2j}} \\ &\le \sum_{j=k+1}^{n} \sigma_j\end{aligned}$$

which lets us conclude that

$$\sigma_k \le \sigma_n + \sum_{j=k+1}^{n} \sigma_j, \quad \text{for } 0 \le k < n.$$

Since $\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{8} \approx 0.78540 + 0.46365 + 0.24498 + 0.12435 \approx 1.618 > \pi/2$, it is also clear that

$$|r| \le \pi/2 < \sum_{j=0}^{3} \tan^{-1} 2^{-j} < \sigma_n + \sum_{j=0}^{n} \sigma_j,$$

which completes the proof. ■

To prove that the CORDIC trigonometric algorithm converges, we first define the sequence $s_k = \theta - z_k = \sum_{j=0}^{k-1} \delta_j \sigma_j$. We see that $s_0 = \theta - z_0 = 0$ and $s_{k+1} = \sum_{j=0}^{k} \delta_j \sigma_j = s_k + \delta_k \sigma_k$. For $r = \theta$, we have $\rho_k = \operatorname{sgn}(r - s_k) = \operatorname{sgn}(\theta - s_k) = \operatorname{sgn}(z_k) = \delta_k$. Hence, this sequence s_k satisfies the Convergence Theorem:

$$|\theta - s_{n+1}| \le \sigma_n = \tan^{-1} 2^{-n} \le 1/2^n.$$

Now we will show that the CORDIC sequences x_k and y_k yield $x_{n+1} \approx \cos(s_{n+1})$ and $y_{n+1} \approx \sin(s_{n+1})$. To this end, we prove the following lemma.

Lemma. *For the sequence s_k defined earlier,*

$$\begin{aligned} \cos(s_{k+1}) &= \cos(\sigma_k)[\cos(s_k) - \delta_k \sin(s_k)\, 2^{-k}] \\ \sin(s_{k+1}) &= \cos(\sigma_k)[\sin(s_k) + \delta_k \cos(s_k) 2^{-k}] \end{aligned}$$

Proof. The proofs are based on elementary trigonometric identities. For the first formula,

$$\begin{aligned} \cos(s_{k+1}) &= \cos(s_k + \delta_k \sigma_k) \\ &= \cos(s_k)\cos(\delta_k \sigma_k) - \sin(s_k)\sin(\delta_k \sigma_k) \\ &= \cos(s_k)\cos(\sigma_k) - \delta_k \sin(s_k)\sin(\sigma_k) \\ &= \cos(\sigma_k)[\cos(s_k) - \delta_k \sin(s_k)\tan(\sigma_k)] \\ &= \cos(\sigma_k)[\cos(s_k) - \delta_k \sin(s_k) 2^{-k}] \end{aligned}$$

Similarly, for the sine formula,

$$\begin{aligned} \sin(s_{k+1}) &= \sin(s_k + \delta_k \sigma_k) \\ &= \sin(s_k)\cos(\delta_k \sigma_k) + \cos(s_k)\sin(\delta_k \sigma_k) \\ &= \sin(s_k)\cos(\sigma_k) + \delta_k \cos(s_k)\sin(\sigma_k) \\ &= \cos(\sigma_k)[\sin(s_k) + \delta_k \cos(s_k)\tan(\sigma_k)] \\ &= \cos(\sigma_k)[\sin(s_k) + \delta_k \cos(s_k) 2^{-k}] \end{aligned}$$

■

Now define

$$
\begin{aligned}
w_0 &= 1/K \\
w_1 &= (\cos\sigma_0)/K \\
&\vdots \\
w_{k+1} &= (\cos\sigma_0\cos\sigma_1\cdots\cos\sigma_k)/K \\
&\vdots \\
w_{n+1} &= (\cos\sigma_0\cos\sigma_1\cdots\cos\sigma_n)/K = 1,
\end{aligned}
$$

and consider the sequences

$$
\begin{aligned}
C_k &= (\cos s_k)/w_k & \qquad C_0 &= (\cos s_0)/w_0 = K \\
S_k &= (\sin s_k)/w_k & \qquad S_0 &= (\sin s_0)/w_0 = 0.
\end{aligned}
$$

Using the lemma we see that

$$
\begin{aligned}
C_{k+1} = \frac{\cos(s_{k+1})}{w_{k+1}} &= \frac{\cos(\sigma_k)[\cos(s_k) - \delta_k \sin(s_k)2^{-k}]}{w_{k+1}} \\
&= \frac{K}{\cos\sigma_0\cdots\cos\sigma_{k-1}}[\cos(s_k) - \delta_k \sin(s_k)2^{-k}] \\
&= \frac{\cos s_k}{w_k} - \frac{\delta_k \sin(s_k)2^{-k}}{w_k} \\
&= C_k - \delta_k S_k 2^{-k}.
\end{aligned}
$$

Similarly,

$$
S_{k+1} = S_k + \delta_k C_k 2^{-k}.
$$

In other words, the equations for C_k and S_k are precisely those of x_k and y_k in the CORDIC algorithm! Since $w_{n+1} = 1$, we have

$$x_{n+1} = C_{n+1} = \cos(s_{n+1})/w_{n+1} = \cos(s_{n+1})$$
$$y_{n+1} = S_{n+1} = \sin(s_{n+1})/w_{n+1} = \sin(s_{n+1}).$$

Finally, the Mean Value Theorem applied to the cosine function says that there exists a number c between θ and s_{n+1} such that

$$\frac{\cos\theta - \cos(s_{n+1})}{\theta - s_{n+1}} = -\sin c.$$

Hence, the inequality $|\theta - s_{n+1}| \leq 1/2^n$ gives us an error bound for the cosine computation.

$$\left|\cos\theta - x_{n+1}\right| = \left|\cos\theta - \cos(s_{n+1})\right| = \left|-\sin c\right|\left|\theta - s_{n+1}\right| \leq \left|\theta - s_{n+1}\right| \leq 1/2^n$$

Similarly, $\left|\sin\theta - y_{n+1}\right| = |\sin\theta - \sin(s_{n+1})| \leq 1/2^n$.

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